NCERT Solutions Class 11 Maths Chapter 13 Limits and Derivatives

Question 1:

Evaluate the given limit: $\lim_{x \to 3} x+3$

Solution:

$$\lim_{x \to 3} x+3 = 3+3$$
$$= 6$$

Question 2:

Evaluate the given limit: $\lim_{x \to \pi}\left(x-\frac{22}{7}\right)$

Solution:

$$\lim_{x \to \pi}\left(x-\frac{22}{7}\right) = \left(\pi-\frac{22}{7}\right)$$

Question 3:

Evaluate the given limit: $\lim_{r \to 1} \pi r^2$

Solution:

$$\lim_{r \to 1} \pi r^2 = \pi(1)^2$$
$$= \pi$$

Question 4:

Evaluate the given limit: $\lim_{x \to 4} \frac{4x+3}{x-2}$

Solution:

$$\lim_{x \to 4} \frac{4x+3}{x-2} = \frac{4(4)+3}{4-2}$$
$$= \frac{16+3}{2}$$
$$= \frac{19}{2}$$

Question 5:

Evaluate the given limit: $\lim\limits_{x \to -1} \dfrac{x^{10}+x^5+1}{x-1}$

Solution:

$$\begin{aligned}\lim_{x \to -1} \frac{x^{10}+x^5+1}{x-1} &= \frac{(-1)^{10}+(-1)^5+1}{-1-1} \\ &= \frac{1-1+1}{-2} \\ &= -\frac{1}{2}a\end{aligned}$$

Question 6:

Evaluate the given limit: $\lim\limits_{x \to 0} \dfrac{(x+1)^5-1}{x}$

Solution:

It is given that, $\lim\limits_{x \to 0} \dfrac{(x+1)^5-1}{x}$

Put $x+1=y$, so that $x=y-1$

Accordingly,

$$\begin{aligned}\lim_{x \to 0} \frac{(x+1)^5-1}{x} &= \lim_{x \to 0} \frac{(y)^5-1}{y-1} \\ &= 5 \times 1^{5-1} \qquad \left[\because \lim_{x \to 0} \frac{x^n-a^n}{x-a} = na^{n-1}\right] \\ &= 5\end{aligned}$$

Hence, $\lim\limits_{x \to 0} \dfrac{(x+1)^5-1}{x} = 5$

Question 7:

Evaluate the given limit: $\lim\limits_{x \to 2} \dfrac{3x^2-x-10}{x^2-4}$

Solution:

At $x=2$, the value of the given rational function takes the form $\dfrac{0}{0}$.

$$\lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(3x+5)}{(x-2)(x+2)}$$
$$=\lim_{x\to 2}\frac{(3x+5)}{(x+2)}$$
$$=\frac{3(2)+5}{2+2}$$
$$=\frac{11}{4}$$

Question 8:

Evaluate the given limit: $\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$

Solution:

At $x=3$, the value of the given rational function takes the form $\frac{0}{0}$.

$$\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)}$$
$$=\lim_{x\to 3}\frac{(x+3)(x^2+9)}{(2x+1)}$$
$$=\frac{(3+3)(3^2+9)}{2(3)+1}$$
$$=\frac{6\times 18}{6+1}$$
$$=\frac{108}{7}$$

Question 9:

Evaluate the given limit: $\lim_{x\to 0}\frac{ax+b}{cx+1}$

Solution:

$$\lim_{x\to 0}\frac{ax+b}{cx+1}=\frac{a(0)+b}{c(0)+1}$$
$$=b$$

Question 10:

Evaluate the Given limit: $\lim_{z\to1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

Solution:

At $z=1$, the value of the given function takes the form $\frac{0}{0}$.

Put $z^{\frac{1}{6}}=x$ so that $z\to1$ as $x\to1$

Accordingly,

$$\lim_{z\to1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}=\lim_{x\to1}\frac{x^2-1}{x-1}$$

$$=\lim_{x\to1}\frac{x^2-1}{x-1}$$

$$=2\times1^{2-1}\qquad\left[\because\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$=2$$

Question 11:

Evaluate the given limit: $\lim_{x\to1}\frac{ax^2+bx+c}{cx^2+bx+a},a+b+c\neq0$

Solution:

$$\lim_{x\to1}\frac{ax^2+bx+c}{cx^2+bx+a}=\frac{a(1)^2+b(1)+c}{c(1)^2+b(1)+a}$$

$$=\frac{a+b+c}{a+b+c}$$

$$=1\qquad[a+b+c\neq0]$$

Question 12:

Evaluate the Given limit: $\lim_{x\to-2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

Solution:

At $x=-2$, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}=\lim_{x\to -2}\frac{\left(\frac{2+x}{2x}\right)}{x+2}$$

$$=\lim_{x\to -2}\frac{1}{2x}$$

$$=\frac{1}{2(-2)}$$

$$=-\frac{1}{4}$$

Question 13:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax}{bx}$

Solution:

$\lim_{x\to 0}\frac{\sin ax}{bx}$

At $x=0$, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{\sin ax}{bx}=\lim_{x\to 0}\frac{\sin ax}{ax}\times\frac{ax}{bx}$$

$$=\lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)\times\frac{a}{b}$$

$$=\frac{a}{b}\lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)\qquad [x\to 0\Rightarrow ax\to 0]$$

$$=\frac{a}{b}\times 1\qquad \left[\because \lim_{y\to 0}\left(\frac{\sin y}{y}\right)=1\right]$$

$$=\frac{a}{b}$$

Question 14:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax}{\sin bx}, a, b\neq 0$

Solution:

At $x=0$, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx}=\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)\times ax}{\left(\frac{\sin bx}{bx}\right)\times bx}$$

$$=\frac{a}{b}\times\frac{\lim\limits_{x\to 0}\left(\frac{\sin ax}{ax}\right)}{\lim\limits_{x\to 0}\left(\frac{\sin bx}{bx}\right)} \qquad \left[\begin{array}{l} x\to 0\Rightarrow ax\to 0 \\ and\ x\to 0\Rightarrow bx\to 0\end{array}\right]$$

$$=\frac{a}{b}\times\frac{1}{1} \qquad \left[\because \lim_{y\to 0}\left(\frac{\sin y}{y}\right)=1\right]$$

$$=\frac{a}{b}$$

Question 15:

Evaluate the Given limit: $\lim\limits_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$

Solution:

It can be seen that $x\to\pi\Rightarrow(\pi-x)\to 0$

Therefore,

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}=\frac{1}{\pi}\lim_{(\pi-x)\to 0}\frac{\sin(\pi-x)}{\pi(\pi-x)}$$

$$=\frac{1}{\pi}\times 1 \qquad \left[\because \lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$=\frac{1}{\pi}$$

Question 16:

Evaluate the Given limit: $\lim\limits_{x\to 0}\frac{\cos x}{\pi-x}$

Solution:

$$\lim_{x\to 0}\frac{\cos x}{\pi - x} = \frac{\cos 0}{\pi - 0}$$
$$= \frac{1}{\pi}$$

Question 17:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\cos 2x - 1}{\cos x - 1}$

Solution 17: $\lim_{x\to 0}\frac{\cos 2x - 1}{\cos x - 1}$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{\cos 2x - 1}{\cos x - 1} = \lim_{x\to 0}\frac{1 - 2\sin^2 x - 1}{1 - 2\sin^2\frac{x}{2} - 1} \qquad \left[\because \cos x = 1 - 2\sin^2\frac{x}{2}\right]$$

$$= \lim_{x\to 0}\frac{\sin^2 x}{\sin^2\frac{x}{2}}$$

$$= \lim_{x\to 0}\frac{\left(\frac{\sin^2 x}{x^2}\right)\times x^2}{\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)\times\frac{x^2}{4}}$$

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=4\frac{\lim\limits_{x\to 0}\left(\frac{\sin^2 x}{x^2}\right)}{\lim\limits_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)}$$

$$=4\frac{\left(\lim\limits_{x\to 0}\frac{\sin x}{x}\right)^2}{\left(\lim\limits_{x\to 0}\frac{\sin\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)^2} \qquad \left[x\to 0\Rightarrow\frac{x}{2}\to 0\right]$$

$$=4\times\frac{1^2}{1^2} \qquad \left[\because \lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$=4$$

Question 18:

Evaluate the given limit: $\lim\limits_{x\to 0}\frac{ax+x\cos x}{b\sin x}$

Solution:

At $x=0$, the value of the given function takes the form $\frac{0}{0}$.
Now,

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}=\frac{1}{b}\lim_{x\to 0}\frac{x(a+\cos x)}{\sin x}$$

$$=\frac{1}{b}\lim_{x\to 0}\left(\frac{x}{\sin x}\right)\times\lim_{x\to 0}(a+\cos x)$$

$$=\frac{1}{b}\times\frac{1}{\left(\lim\limits_{x\to 0}\frac{\sin x}{x}\right)}\times\lim_{x\to 0}(a+\cos x)$$

$$=\frac{1}{b}\times(a+\cos 0) \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=\frac{a+1}{b}$$

Question 19:

Evaluate the Given limit: $\lim_{x \to 0} x \sec x$

Solution:

$$\begin{aligned}\lim_{x \to 0} x \sec x &= \lim_{x \to 0} \frac{x}{\cos x} \\ &= \frac{0}{\cos 0} \\ &= \frac{0}{1} \\ &= 0\end{aligned}$$

Question 20:

Evaluate the Given limit: $\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx} a, b, a + b \neq 0$

Solution:

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\begin{aligned}\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx} &= \lim_{x \to 0} \frac{\left(\frac{\sin ax}{ax}\right) ax + bx}{ax + bx\left(\frac{\sin bx}{bx}\right)} \\ &= \frac{\left(\lim_{x \to 0} \frac{\sin ax}{ax}\right) \times \lim_{x \to 0}(ax) + \lim_{x \to 0}(bx)}{\lim_{x \to 0} ax + \lim_{x \to 0} bx\left(\lim_{x \to 0} \frac{\sin bx}{bx}\right)} && [\text{As } x \to 0 \Rightarrow ax \to 0 \text{ and bx} \to 0] \\ &= \frac{\lim_{x \to 0}(ax) + \lim_{x \to 0} bx}{\lim_{x \to 0} ax + \lim_{x \to 0} bx} && \left[\lim_{x \to 0} \frac{\sin x}{x} = 1\right] \\ &= \frac{\lim_{x \to 0}(ax + bx)}{\lim_{x \to 0}(ax + bx)} \\ &= \lim_{x \to 0}(1) \\ &= 1\end{aligned}$$

Question 21:

Evaluate the Given limit: $\lim_{x \to 0}(\operatorname{cosec} x - \cot x)$

Solution:

At $x=0$, the value of the given function takes the form $\infty-\infty$

Now,

$$\lim_{x\to 0}(\operatorname{cosec} x-\cot x)=\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right)$$
$$=\lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right)$$
$$=\lim_{x\to 0}\frac{\left(\frac{1-\cos x}{x}\right)}{\left(\frac{\sin x}{x}\right)}$$
$$=\frac{\lim\limits_{x\to 0}\frac{1-\cos x}{x}}{\lim\limits_{x\to 0}\frac{\sin x}{x}}$$
$$=\frac{0}{1} \qquad \left[\because \lim_{x\to 0}\frac{1-\cos x}{x}=0 \text{ and } \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$
$$=0$$

Question 22:

Evaluate the Given limit: $\lim\limits_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$

Solution:

At $x=\frac{\pi}{2}$, the value of the given function takes the form $\frac{0}{0}$.

Now, put $x-\frac{\pi}{2}=y$ so that $x\to\frac{\pi}{2}, y\to 0$

Therefore,

$$\lim_{x\to\frac{\pi}{2}} \frac{\tan 2x}{x-\frac{\pi}{2}} = \lim_{y\to 0} \frac{\tan 2\left(y+\frac{\pi}{2}\right)}{y}$$

$$= \lim_{y\to 0} \frac{\tan(\pi+2y)}{y}$$

$$= \lim_{y\to 0} \frac{\tan 2y}{y} \qquad [\because \tan(\pi+2y) = \tan 2y]$$

$$= \lim_{y\to 0} \frac{\sin 2y}{y\cos 2y}$$

$$= \lim_{y\to 0} \left(\frac{\sin 2y}{2y} \times \frac{2}{\cos 2y}\right)$$

$$= \left(\lim_{y\to 0} \frac{\sin 2y}{2y}\right) \times \lim_{y\to 0}\left(\frac{2}{\cos 2y}\right) \qquad [y\to 0 \Rightarrow 2y \to 0]$$

$$\lim_{x\to\frac{\pi}{2}} \frac{\tan 2x}{x-\frac{\pi}{2}} = 1\times\frac{2}{\cos 0} \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$= 1\times\frac{2}{1}$$

$$= 2$$

Question 23:

Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} 2x+3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$

Solution:

The given function is $f(x) = \begin{cases} 2x+3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$

Now,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}[2x+3]$$
$$= 2(0)+3$$
$$= 3$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} 3(x+1)$$
$$= 3(0+1)$$
$$= 3$$

Hence,
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x) = 3$$

Now,
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(x+1)$$
$$= 3(1+1)$$
$$= 6$$

Hence,
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1} f(x) = 6$$

Question 24:

Find $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} x^2-1, & x \le 1 \\ -x-1, & x>1 \end{cases}$

Solution:

The given function is $f(x) = \begin{cases} x^2-1, & x \le 1 \\ -x-1, & x>1 \end{cases}$

Now,
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}\left[x^2-1\right]$$
$$= 1^2-1$$
$$= 1-1$$
$$= 0$$

It is observed that $\lim_{x\to 1^-} f(x) \ne \lim_{x\to 1} f(x)$

Hence, $\lim_{x \to 1} f(x)$ does not exist.

Question 25:

Evaluate $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution:

The given function is $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Now,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left[\frac{|x|}{x}\right] = \lim_{x \to 0} \left(\frac{-x}{x}\right) \qquad [\text{When } x \text{ is negative, } |x| = -x]$$
$$= \lim_{x \to 0} (-1)$$
$$= -1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left[\frac{|x|}{x}\right]$$
$$= \lim_{x \to 0} \left(\frac{x}{x}\right) \qquad [\text{When } x \text{ is positive, } |x| = x]$$
$$= \lim_{x \to 0} (1)$$
$$= 1$$

It is observed that $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

Hence, $\lim_{x \to 0} f(x)$ does not exist.

Question 26:

Find $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution:

The given function is $f(x)=\begin{cases}\dfrac{x}{|x|}, & x\neq 0\\ 0, & x=0\end{cases}$

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-}\left[\frac{x}{|x|}\right]\\ &=\lim_{x\to 0}\left(\frac{x}{-x}\right) \qquad \left[\text{when } x<0, |x|=-x\right]\\ &=\lim_{x\to 0}(-1)\\ &=-1\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}\left[\frac{x}{|x|}\right]\\ &=\lim_{x\to 0}\left(\frac{x}{x}\right) \qquad \left[\text{when } x>0, |x|=x\right]\\ &=\lim_{x\to 0}(1)\\ &=1\end{aligned}$$

It is observed that $\lim\limits_{x\to 0^-} f(x)\neq \lim\limits_{x\to 0^+} f(x)$.

Hence, $\lim\limits_{x\to 0} f(x)$ does not exist.

Question 27:

Find $\lim\limits_{x\to 5} f(x)$, where $f(x)=|x|-5$

Solution:

The given function is $f(x)=|x|-5$

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} \left(|x| - 5\right)$$
$$= \lim_{x \to 5} (x - 5) \qquad \left[\text{when } x > 0, |x| = x\right]$$
$$= 5 - 5$$
$$= 0$$

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} \left(|x| - 5\right)$$
$$= \lim_{x \to 5} (x - 5) \qquad \left[\text{when } x > 0, |x| = x\right]$$
$$= 5 - 5$$
$$= 0$$

Therefore,

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = 0$$

Hence, $\lim_{x \to 5} f(x) = 0$.

Question 28:

Suppose $f(x) = \begin{cases} a - bx, & x < 0 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$ and if $\lim_{x \to 1} f(x) = f(1)$ what are possible values of a and b?

Solution:

The given function is $f(x) = \begin{cases} a - bx, & x < 0 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$

Therefore,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (a + bx)$$
$$= a + b$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (b - ax)$$
$$= b - a$$

$$f(1) = 4$$

It is given that $\lim_{x \to 1} f(x) = f(1)$

Therefore, $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1} f(x) = f(1)$

Hence, $a+b=4$ and $b-a=4$

On solving these two equations, we obtain

$a=0$ and $b=4$

Thus, the respective possible values of *a* and *b* are 0 and 4.

Question 29:

Let $a_1, a_2, \dots, a_n$ be fixed real numbers and define a function

$f(x)=(x-a_1)(x-a_2)\dots(x-a_n)$

What is $\lim\limits_{x\to a_1} f(x)$? For some $a \neq a_1, a_2, \dots, a_n$, compute $\lim\limits_{x\to a} f(x)$

Solution:

The given function is $f(x)=(x-a_1)(x-a_2)\dots(x-a_n)$

$$\begin{aligned}\lim_{x\to a_1} f(x) &= \lim_{x\to a_1}\left[(x-a_1)(x-a_2)\dots(x-a_n)\right]\\ &= (a_1-a_1)(a_1-a_2)\dots(a_1-a_n)\end{aligned}$$

$$\lim_{x\to a_1} f(x) = 0$$

Now,

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\left[(x-a_1)(x-a_2)\dots(x-a_n)\right]\\ &= (a-a_1)(a-a_2)\dots(a-a_n)\end{aligned}$$

Thus, $\lim\limits_{x\to a} f(x) = (a-a_1)(a-a_2)\dots(a-a_n)$

Question 30:

If $f(x)=\begin{cases}|x|+1, & x<0\\ 0, & x=0\\ |x|-1, & x>1\end{cases}$

For what value (s) of does $\lim\limits_{x\to a} f(x)$ exists?

Solution:

The given function is $\text{If } f(x)=\begin{cases}|x|+1, & x<0\\ 0, & x=0\\ |x|-1, & x>1\end{cases}$

When $a=0$

$$\lim_{x\to0^-} f(x)=\lim_{x\to0^-} f(|x|+1)$$
$$=\lim_{x\to0}(-x+1) \qquad [If\ x<0, |x|=-x]$$
$$=0+1$$
$$=1$$

$$\lim_{x\to0^+} f(x)=\lim_{x\to0^+}(|x|-1)$$
$$=\lim_{x\to0}(x-1) \qquad [If\ x>0, |x|=x]$$
$$=0-1$$
$$=-1$$

Here it is observed that $\lim_{x\to0^-} f(x)\neq\lim_{x\to0^+} f(x)$

Therefore, $\lim_{x\to0} f(x)$ does not exist at $x=a$ where $a=0$.

When $a<0$

$$\lim_{x\to a^-} f(x)=\lim_{x\to a^-}(|x|+1)$$
$$=\lim_{x\to a}(-x+1) \qquad [x<a<0\Rightarrow|x|=-x]$$
$$=-a+1$$

$$\lim_{x\to a^+} f(x)=\lim_{x\to a^+}(|x|+1)$$
$$=\lim_{x\to a}(-x+1) \qquad [a<x<0\Rightarrow|x|=-x]$$
$$=-a+1$$

Here, $\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=-a+1$

Thus, limit of $f(x)$ exists at $x=a$ where $a<0$

When $a>0$

$$\begin{aligned}\lim_{x\to a^-} f(x) &= \lim_{x\to a^-}(|x|+1)\\ &= \lim_{x\to a}(x-1) \qquad [0<x<a \Rightarrow |x|=x]\\ &= a-1\end{aligned}$$

$$\begin{aligned}\lim_{x\to a^+} f(x) &= \lim_{x\to a^+}(|x|-1)\\ &= \lim_{x\to a}(x-1) \qquad [0<x<a \Rightarrow |x|=x]\\ &= a-1\end{aligned}$$

Here, $\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=a-1$

Thus, limit of $f(x)$ exists at $x=a$ where $a>0$

Hence, $\lim_{x\to a} f(x)$ exists for all $a\neq 0$.

Question 31:

If the function $f(x)$ satisfies $\lim_{x\to 1}\frac{f(x)-2}{x^2-1}=\pi$, evaluate $\lim_{x\to 1} f(x)$.

Solution:

It is given that the function $f(x)$ satisfies $\lim_{x\to 1}\frac{f(x)-2}{x^2-1}=\pi$,

$$\lim_{x\to 1}\frac{f(x)-2}{x^2-1}=\frac{\lim_{x\to 1}(f(x)-2)}{\lim_{x\to 1}(x^2-1)}=\pi$$

$$\begin{aligned}&\Rightarrow \lim_{x\to 1}(f(x)-2)=\pi\lim_{x\to 1}(x^2-1)\\ &\Rightarrow \lim_{x\to 1}(f(x)-2)=\pi(1^2-1)\\ &\Rightarrow \lim_{x\to 1}(f(x)-2)=0\\ &\Rightarrow \lim_{x\to 1} f(x)-\lim_{x\to 1}2=0\\ &\Rightarrow \lim_{x\to 1} f(x)-2=0\\ &\Rightarrow \lim_{x\to 1} f(x)=2\end{aligned}$$

Hence, $\lim_{x\to 1} f(x)=2$

Question 32:

If $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$. For what integers m and n does both $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exists?

Solution:

It is given that $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$

Therefore,

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0}\left(mx^2+n\right) \\ &= m(0)^2+n \\ &= n\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0}(nx+m) \\ &= n(0)+m \\ &= m\end{aligned}$$

Thus, $\lim_{x\to 0^+} f(x)$ exists if $m=n$

Now,

$$\begin{aligned}\lim_{x\to 1^-} f(x) &= \lim_{x\to 1}(nx+m) \\ &= n(1)+m \\ &= m+n\end{aligned}$$

$$\begin{aligned}\lim_{x\to 1^+} f(x) &= \lim_{x\to 1}\left(nx^3+m\right) \\ &= n(1)^3+m \\ &= m+n\end{aligned}$$

Therefore, $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=\lim_{x\to 1} f(x)$

Thus, $\lim_{x\to 1} f(x)$ exists for any integral value of m and n.

EXERCISE 13.2

Question 1:

Find the derivative of $x^2 - 2$ at $x = 10$.

Solution:

Let $f(x) = x^2 - 2$

Accordingly,

$$f'(10) = \lim_{h \to 0} \frac{f(10+h) - f(10)}{h}$$

$$= \lim_{h \to 0} \frac{\left[(10+h)^2 - 2\right] - (10^2 - 2)}{h}$$

$$= \lim_{h \to 0} \frac{10^2 + 2.10.h + h^2 - 2 - 10^2 + 2}{h}$$

$$= \lim_{h \to 0} \frac{20h + h^2}{h}$$

$$= \lim_{h \to 0} (20 + h)$$

$$= 20 + 0$$

$$= 20$$

Thus, the derivative of $x^2 - 2$ at $x = 10$ is 20.

Question 2:

Find the derivative of $99x$ at $x = 100$.

Solution:

Let $f(x) = 99x$

Accordingly,

$$\begin{aligned}f'(100)&=\lim_{h\to 0}\frac{f(100+h)-f(100)}{h}\\&=\lim_{h\to 0}\frac{99(100+h)-99(100)}{h}\\&=\lim_{h\to 0}\frac{99\times 100+99h-99\times 100}{h}\\&=\lim_{h\to 0}\frac{99h}{h}\\&=\lim_{h\to 0}(99)\\&=99\end{aligned}$$

Thus, the derivative of $99x$ at $x=100$ is 99.

Question 3:

Find the derivative of x at $x=1$.

Solution:

Let $f(x)=x$

Accordingly,

$$\begin{aligned}f'(1)&=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}\\&=\lim_{h\to 0}\frac{(1+h)-1}{h}\\&=\lim_{h\to 0}\frac{h}{h}\\&=\lim_{h\to 0}(1)\\&=1\end{aligned}$$

Thus, the derivative of x at $x=1$ is 1.

Question 4:

Find the derivative of the following functions from first principle.

(i) x^3-27

(ii) $(x-1)(x-2)$

(iii) $\frac{1}{x^2}$

(iv) $\frac{x+1}{x-1}$

Solution:

(i) Let $f(x)=x^3-27$

Accordingly, from the first principle

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\left[(x+h)^3-27\right]-\left(x^3-27\right)}{h}$$
$$=\lim_{h\to 0}\frac{x^3+h^3+3x^2h+3xh^2-x^3}{h}$$
$$=\lim_{h\to 0}\frac{h^3+3x^2h+3xh^2}{h}$$
$$=\lim_{h\to 0}\left(h^3+3x^2h+3xh^2\right)$$
$$=0+3x^2+0$$
$$=3x^2$$

(ii) Let $f(x)=(x-1)(x-2)$

Accordingly, from the first principle

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{(x+h-1)(x+h-2)-(x-1)(x-2)}{h}$$
$$=\lim_{h\to 0}\frac{\left(x^2+hx-2x+hx+h^2-2h-x-h+2\right)-\left(x^2-2x-x+2\right)}{h}$$
$$=\lim_{h\to 0}\frac{2hx+h^2-3h}{h}$$
$$=\lim_{h\to 0}(2x+h-3)$$
$$=2x-3$$

(iii) Let $f(x)=\frac{1}{x^2}$

Accordingly, from the first principle

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\
&= \lim_{h \to 0} \frac{1}{h}\left[\frac{x^2 - (x+h)^2}{x^2(x+h)^2}\right] \\
&= \lim_{h \to 0} \frac{1}{h}\left[\frac{x^2 - x^2 - 2hx - h^2}{x^2(x+h)^2}\right] \\
&= \lim_{h \to 0} \frac{1}{h}\left[\frac{-h^2 - 2hx}{x^2(x+h)^2}\right] \\
&= \lim_{h \to 0}\left[\frac{-h - 2x}{x^2(x+h)^2}\right] \\
&= \frac{0 - 2x}{x^2(x+0)^2} \\
&= \frac{-2}{x^3}
\end{aligned}
$$

(iv) Let $f(x) = \dfrac{x+1}{x-1}$

Accordingly, from the first principle

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\left(\frac{x+h+1}{x+h-1}-\frac{x+1}{x-1}\right)}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{(x-1)(x+h-1)}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x^2+hx+x-x-h-1)-(x^2+hx-x+x-h-1)}{(x-1)(x+h-1)}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-2h}{(x-1)(x+h-1)}\right]$$
$$=\lim_{h\to 0}\left[\frac{-2}{(x-1)(x+h-1)}\right]$$
$$=\frac{-2}{(x-1)(x-1)}$$
$$=\frac{-2}{(x-1)^2}$$

Question 5:

For the function

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1$$

Prove that $f'(1)=100f'(0)$

Solution:

The given function is $f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left[\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1\right]$$
$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(\frac{x^{100}}{100}\right)+\frac{d}{dx}\left(\frac{x^{99}}{99}\right)+\ldots+\frac{d}{dx}\left(\frac{x^2}{2}\right)+\frac{d}{dx}(x)+\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}(x^n)=nx^{n-1}$, we obtain

$$\frac{d}{dx}f(x)=\frac{100x^{99}}{100}+\frac{99x^{98}}{99}+\ldots+\frac{2x}{2}+1+0$$
$$=x^{99}+x^{98}+\ldots+x+1$$

Therefore, $f'(x)=x^{99}+x^{98}+\ldots\ldots+x+1$

At $x=0$,

$f'(0)=1$

At $x=1$,

$$f'(1)=1^{99}+1^{98}+\ldots+1+1$$
$$=[1+1+\ldots..+1+1]_{100\ terms}$$
$$=1\times 100$$
$$=100f'(0)$$

Thus, $f'(1)=100f'(0)$

Question 6:

Find the derivative of $x^n+ax^{n-1}+a^2x^{n-2}+\ldots+a^{n-1}x+a^n$ for some fixed real number a.

Solution:

Let $f(x)=x^n+ax^{n-1}+a^2x^{n-2}+\ldots+a^{n-1}x+a^n$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(x^n+ax^{n-1}+a^2x^{n-2}+\ldots+a^{n-1}x+a^n\right)$$
$$=\frac{d}{dx}(x^n)+a\frac{d}{dx}(x^{n-1})+a^2v(x^{n-2})+\ldots+a^{n-1}\frac{d}{dx}(x)+a^n\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}(x^n)=nx^{n-1}$, we obtain

$$f'(x)=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+\ldots+a^{n-1}+a^n(0)$$

Thus, $f'(x)=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+\ldots+a^{n-1}$

Question 7:

For some constants *a* and *b*, find the derivative of

(i) $(x-a)(x-b)$

(ii) $(ax^2+b)^2$

(iii) $\dfrac{x-a}{x-b}$

Solution:

(i) Let $f(x)=(x-a)(x-b)$

Therefore,

$$f(x)=x^2-(a+b)x+ab$$
$$=\frac{d}{dx}\left[x^2-(a+b)x+ab\right]$$
$$=\frac{d}{dx}\left(x^2\right)-(a+b)\frac{dy}{dx}(x)+\frac{d}{dx}(ab)$$

On using theorem $\dfrac{d}{dx}\left(x^n\right)=nx^{n-1}$, we obtain

$$f(x)=2x-(a+b)+0$$
$$=2x-a-b$$

(ii) Let $f(x)=\left(ax^2+b\right)^2$

Therefore,

$$f(x)=a^2x^4+2abx^2+b^2$$
$$=\frac{d}{dx}\left(a^2x^4+2abx^2+b^2\right)$$
$$=a^2\frac{d}{dx}\left(x^4\right)+2ab\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}b^2$$

On using theorem $\dfrac{d}{dx}\left(x^n\right)=nx^{n-1}$, we obtain

$$f(x)=a^2\left(4x^3\right)+2ab(2x)+b^2(0)$$
$$=4a^2x^3+4abx$$
$$=4ax\left(ax^2+b\right)$$

(iii) Let $f(x)=\dfrac{x-a}{x-b}$

Therefore,

$$f(x)=\frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

By quotient rule

$$f(x)=\frac{(x-b)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$=\frac{(x-b)(1)-(x-a)(1)}{(x-b)^2}$$

$$=\frac{x-b-x+a}{(x-b)^2}$$

$$=\frac{a-b}{(x-b)^2}$$

Question 8:

Find the derivative of $\frac{x^n-a^n}{x-a}$ for some constant a.

Solution:

Let $f(x)=\frac{x^n-a^n}{x-a}$

$$f'(x)=\frac{d}{dx}\left(\frac{x^n-a^n}{x-a}\right)$$

By quotient rule,

$$f'(x)=\frac{(x-a)\frac{d}{dx}(x^n-a^n)-(x^n-a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$$

$$=\frac{(x-a)(nx^{n-1}-0)-(x^n-a^n)}{(x-a)^2}$$

$$=\frac{nx^n-anx^{n-1}-x^n+a^n}{(x-a)^2}$$

Question 9:

Find the derivative of

(i) $2x-\frac{3}{4}$

(ii) $\left(5x^3+3x-1\right)(x-1)$

(iii) $x^{-3}(5+3x)$

(iv) $x^5\left(3-6x^{-9}\right)$

(v) $x^{-4}\left(3-4x^{-5}\right)$

(vi) $\frac{2}{x+1}-\frac{x^2}{3x-1}$

Solution:

(i) Let $f(x)=2x-\frac{3}{4}$

$$f'(x)=\frac{d}{dx}\left(2x-\frac{3}{4}\right)$$
$$=2\frac{d}{dx}(x)-\frac{d}{dx}\left(\frac{3}{4}\right)$$
$$=2-0$$
$$=2$$

(ii) Let $f(x)=\left(5x^3+3x-1\right)(x-1)$

By Leibnitz product rule,

$$f'(x)=\left(5x^3+3x-1\right)\frac{d}{dx}(x-1)+(x-1)\frac{d}{dx}\left(5x^3+3x-1\right)$$
$$=\left(5x^3+3x-1\right)(1)+(x-1)\left(5.3x^2+3-0\right)$$
$$=\left(5x^3+3x-1\right)+(x-1)\left(15x^2+3\right)$$
$$=5x^3+3x-1+15x^3+3x-15x^2-3$$
$$=20x^3-15x^2+6x-4$$

(iii) Let $f(x)=x^{-3}(5+3x)$

By Leibnitz product rule,

$$f'(x) = x^{-3}\frac{d}{dx}(5+3x) + (5+3x)\frac{d}{dx}(x^{-3})$$
$$= x^{-3}(0+3) + (5+3x)(-3x^{-3-1})$$
$$= x^{-3}(3) + (5+3x)(-3x^{-4})$$
$$= 3x^{-3} - 15x^{-4} - 9x^{-3}$$
$$= -6x^{-3} - 15x^{-4}$$
$$= -3x^{-3}\left(2+\frac{5}{x}\right)$$
$$= \frac{-3x^{-3}}{x}(2x+5)$$
$$= \frac{-3}{x^4}(5+2x)$$

(iv) Let $f(x) = x^5(3-6x^{-9})$

By Leibnitz product rule,

$$f'(x) = x^5\frac{d}{dx}(3-6x^{-9}) + (3-6x^{-9})\frac{d}{dx}(x^5)$$
$$= x^5\{0-6(-9)x^{-9-1}\} + (3-6x^{-9})(5x^4)$$
$$= x^5(54x^{-10}) + 15x^4 - 30x^{-5}$$
$$= 24x^{-5} + 15x^4$$
$$= 15x^4 + \frac{24}{x^5}$$

(v) Let $f(x) = x^{-4}(3-4x^{-5})$

By Leibnitz product rule,

$$f'(x) = x^{-4}\frac{d}{dx}(3-4x^{-5}) + (3-4x^{-5})\frac{d}{dx}(x^{-4})$$
$$= x^{-4}\{0-4(-5)x^{-5-1}\} + (3-4x^{-5})(-4)x^{-4-1}$$
$$= x^{-4}(20x^{-6}) + (3-4x^{-5})(-4x^{-5})$$
$$= 20x^{-10} - 12x^{-5} + 16x^{-10}$$
$$= 36x^{-10} - 12x^{-5}$$
$$= \frac{12}{x^5} + \frac{36}{x^{10}}$$

(vi) Let $f(x) = \frac{2}{x+1} - \frac{x^2}{3x-1}$

$$f'(x) = \frac{d}{dx}\left(\frac{2}{x+1}\right) - \frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$$

By quotient rule,

$$f'(x)=\left[\frac{(x+1)\frac{d}{dx}(2)-2\frac{d}{dx}(x+1)}{(x+1)^2}\right]-\left[\frac{(3x-1)\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$

$$=\left[\frac{(x+1)(0)-2(1)}{(x+1)^2}\right]-\left[\frac{(3x-1)(2x)-x^2(3)}{(3x-1)^2}\right]$$

$$=\frac{-2}{(x+1)^2}-\left[\frac{6x^2-2x-3x^2}{(3x-1)^2}\right]$$

$$=\frac{-2}{(x+1)^2}-\frac{x(3x-2)}{(3x-1)^2}$$

Question 10:

Find the derivative of $\cos x$ from first principle.

Solution:

Let $f(x)=\cos x$.

Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\left[\frac{\cos(x+h)-\cos(x)}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h}\frac{-\sin x\sin h}{h}\right]$$

$$=-\cos x\left[\lim_{h\to 0}\left(\frac{1-\cos h}{h}\right)\right]-\sin x\left[\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\right]$$

$$=-\cos x(0)-\sin x(1) \qquad \left[\because \lim_{h\to 0}\frac{1-\cosh}{h}=0 \text{ and } \lim_{h\to 0}\frac{\sinh}{h}=1\right]$$

$$=-\sin x$$

Question 11:

Find the derivative of following functions:

(i) $\sin x\cos x$

(ii) $\sec x$

(iii) $5\sec x+4\cos x$

(iv) $\operatorname{cosec} x$

(v) $3\cot x+5\operatorname{cosec} x$

(vi) $5\sin x-6\cos x+7$

(vii) $2\tan x-7\sec x$

Solution:

(i) Let $f(x)=\sin x\cos x$.

Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sin(x+h)\cos(x+h)-\sin x\cos x}{h}$$

$$=\lim_{h\to 0}\frac{1}{2h}\left[2\sin(x+h)\cos(x+h)-2\sin x\cos x\right]$$

$$=\lim_{h\to 0}\frac{1}{2h}\left[\sin 2(x+h)-\sin 2x\right]$$

$$=\lim_{h\to 0}\frac{1}{2h}\left[2\cos\frac{2x+2h+2x}{2}.\sin\frac{2x+2h-2x}{h}\right]$$

$$=\lim_{h\to 0}\frac{1}{2h}\left[2\cos\frac{4x+2h}{2}.\sin\frac{2h}{2}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\cos(2x+h)\sin h\right]$$

$$f'(x)=\lim_{h\to 0}\cos(2x+h).\lim_{h\to 0}\frac{\sin h}{h}$$

$$=\cos(2x+h).1$$

$$=\cos 2x$$

(ii) Let $f(x)=\sec x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\sec(x+h)-\sec x}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$
$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$
$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\cos(x+h)}\right]$$
$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{2h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)}}{\cos(x+h)}\right]$$
$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}.\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}$$
$$=\frac{1}{\cos x}.1.\frac{\sin x}{\cos x}$$
$$=\sec x\tan x$$

(iii) Let $f(x)=5\sec x+4\cos x$.
Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{5\sec(x+h)+4\cos(x+h)-[5\sec x+4\cos x]}{h}$$

$$=5\lim_{h\to 0}\frac{[\sec(x+h)-\sec x]}{h}+4\lim_{h\to 0}\frac{[\cos(x+h)-\cos x]}{h}$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]+4\lim_{h\to 0}\frac{1}{h}[\cos x\cos h-\sin x\sin h-\cos x]$$

$$=\frac{5}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\cos(x+h)}\right]+4\left[-\cos\lim_{h\to 0}\frac{(1-\cos x)}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}\right]$$

$$=\frac{5}{\cos x}.\lim_{h\to 0}\frac{\left[\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)}\right]}{\cos(x+h)}+4[-\cos x(0)-\sin x(1)]$$

$$=\frac{5}{\cos x}.\lim_{h\to 0}\left[\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}.\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}\right]-4\sin x$$

$$=\frac{5}{\cos x}.\frac{\sin x}{\cos x}.1-4\sin x$$

$$=5\sec x\tan x-4\sin x$$

(iv) Let $f(x)=\operatorname{cosec} x$.

Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}[\operatorname{cosec}(x+h)-\operatorname{cosec} x]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right).\sin\left(\frac{x-x-h}{2}\right)}{\sin x\sin(x+h)}\right]$$

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right).\sin\left(\frac{-h}{2}\right)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{\left[-\cos\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)}\right]}{\sin x\sin(x+h)}$$

$$\lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin x\sin(x+h)}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$=-\text{cosec}\, x\cot x$$

(v) Let $f(x)=3\cot x+5\ \text{cosec}\ x.$

Accordingly, from the first principle

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[3\cot(x+h)+5\ \text{cosec}(x+h)-3\cot x-5\ \text{cosec}\ x\right]$$

$$=3\lim_{h\to 0}\frac{1}{h}\left[\cot(x+h)-\cot x\right]+5\lim_{h\to 0}\frac{1}{h}\left[\text{cosec}(x+h)-\text{cosec}\ x\right] \qquad \ldots(1)$$

Now,

$$\lim_{h\to 0}\frac{1}{h}\left[\cot(x+h)-\cot x\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{-\sinh}{h}.\lim_{h\to 0}\left[\frac{1}{\sin x\sin(x+h)}\right]$$

$$=-1.\frac{1}{\sin x\sin(x+0)}=\frac{-1}{\sin^2 x}=-\operatorname{cosec}^2 x \qquad \ldots(2)$$

$$\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec} x\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right).\sin\left(\frac{-h}{2}\right)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right).\sin\left(\frac{-h}{2}\right)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{\cos\left(\frac{2x+h}{2}\right).\frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin x\sin(x+h)}$$

$$=\lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin x\sin(x+h)}\right).\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$=-\operatorname{cosec} x\cot x \qquad \ldots(3)$$

From (1), (2) and (3), we obtain

$$f'(x)=-3\operatorname{cosec}^2 x-5\operatorname{cosec} x\cot x$$

(vi) Let $f(x)=5\sin x-6\cos x+7$

Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[5\sin(x+h)-6\cos(x+h)+7-5\sin x+6\cos x-7\right]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h)-\sin x\right]-6\lim_{h\to 0}\frac{1}{h}\left[\cos(x+h)-\cos x\right]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+x}{2}\right).\sin\left(\frac{x+h-x}{2}\right)\right]-6\lim_{h\to 0}\left[\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\right]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h}{2}\right).\sin\left(\frac{h}{2}\right)\right]-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)-\sin x\sin h}{h}\right]$$

$$f'(x)=5\lim_{h\to 0}\frac{1}{h}\left[\cos\left(\frac{2x+h}{2}\right).\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h}-\frac{\sin x\sin h}{h}\right]$$

$$=5\left[\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right]\left[\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]-6\left[-\cos x\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right)-\sin x\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\right]$$

$$=5\cos x.1-6\left[(-\cos x).(0)-\sin x.1\right]$$

$$=5\cos x+6\sin x$$

(vii) Let $f(x)=2\tan x-7\sec x.$

Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[2\tan(x+h)-7\sec(x+h)-2\tan x+7\sec x\right] \\
&= 2\lim_{h\to 0}\frac{1}{h}\left[\tan(x+h)-\tan x\right]-7\lim_{h\to 0}\frac{1}{h}\left[\sec(x+h)-\sec x\right] \\
&= 2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right] \\
&= 2\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x\sin(x+h)-\sin x\cos(x+h)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right] \\
&= 2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x+h-x}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos x\cos(x+h)}\right] \\
&= 2\left[\lim_{h\to 0}\left(\frac{\sinh}{h}\right)\frac{1}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\cos x\cos(x+h)}\right] \\
&= 2\left(\lim_{h\to 0}\frac{\sinh}{h}\right)\left[\lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}\right]-7\left(\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right)\left(\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x\cos(x+h)}\right) \\
&= 2\times 1\times 1\frac{1}{\cos x\cos x}-7\times 1\left(\frac{\sin x}{\cos x\cos x}\right) \\
&= 2\sec^2 x-7\sec x\tan x
\end{aligned}
$$

MISCELLANEOUS EXERCISE

Question 1:

Find the derivative of the following functions from first principle:

(i) $-x$

(ii) $(-x)^{-1}$

(iii) $\sin(x+1)$

(iv) $\cos\left(x-\frac{\pi}{8}\right)$

Solution:

(i) Let $f(x)=-x$

Accordingly, $f(x+h)=-(x+h)$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{-(x+h)-(-x)}{h}$$
$$=\lim_{h\to 0}\frac{-x-h+x}{h}$$
$$=\lim_{h\to 0}\frac{-h}{h}$$
$$=\lim_{h\to 0}(-1)$$
$$=-1$$

(ii) Let $f(x)=(-x)^{-1}=\left(\frac{1}{-x}\right)=\frac{-1}{x}$

Accordingly, $f(x+h)=\frac{-1}{(x+h)}$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-1}{(x+h)}-\left(\frac{-1}{x}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-x+(x+h)}{x(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{x(x+h)}$$

$$=\frac{1}{x.x}$$

$$=\frac{1}{x^2}$$

(iii) Let $f(x)=\sin(x+1)$

Accordingly, $f(x+h)=\sin(x+h+1)$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h+1)-\sin(x+1)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+1+x+1}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h+2}{2}\right).\sin\left(\frac{h}{2}\right)\right]$$

$$=\lim_{h\to 0}\left[\cos\left(\frac{2x+h+2}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

$$=\lim_{h\to 0}\cos\left(\frac{2x+h+2}{2}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \qquad \left[\text{As } h\to 0\Rightarrow\frac{h}{2}\to 0\right]$$

$$=\cos\left(\frac{2x+0+2}{2}\right).1 \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=\cos(x+1)$$

(iv) Let $f(x)=\cos\left(x-\frac{\pi}{8}\right)$

Accordingly, $f(x+h)=\cos\left(x+h-\frac{\pi}{8}\right)$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\cos\left(x+h-\frac{\pi}{8}\right)-\cos\left(x-\frac{\pi}{8}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[-2\sin\frac{\left(x+h-\frac{\pi}{8}+x-\frac{\pi}{8}\right)}{2}\sin\left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[-2\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right).\sin\left(\frac{h}{2}\right)\right]$$

$$f'(x)=\lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

$$=\lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\right].\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\quad\left[\text{As } h\to 0\Rightarrow\frac{h}{2}\to 0\right]$$

$$=\ -\sin\left(\frac{2x+0-\frac{\pi}{4}}{2}\right).1$$

$$=\ -\sin\left(x-\frac{\pi}{8}\right)$$

Question 2:

Find the derivative of the following function (it is to be understood that a is fixed non-zero constant): $(x+a)$

Solution:

Let $f(x)=(x+a)$

Accordingly, $f(x+h)=x+h+a$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{x+h+a-x-a}{h}$$
$$=\lim_{h\to 0}\left(\frac{h}{h}\right)$$
$$=\lim_{h\to 0}(1)$$
$$=1$$

Question 3:

Find the derivative of the following functions (it is to be understood that p, q, r and s are fixed non-zero constants): $(px+q)\left(\frac{r}{x}+s\right)$

Solution:

Let $f(x)=(px+q)\left(\frac{r}{x}+s\right)$

By Leibnitz product rule,

$$f\ '(x)=(px+q)\left(\frac{r}{x}+s\right)'+\left(\frac{r}{x}+s\right)(px+q)'$$
$$=(px+q)(rx^{-1}+s)'+\left(\frac{r}{x}+s\right)(p)$$
$$=(px+q)(-rx^{-2})+\left(\frac{r}{x}+s\right)p$$
$$=(px+q)\left(\frac{-r}{x^2}\right)+\left(\frac{r}{x}+s\right)p$$
$$=\frac{-pr}{x}-\frac{qr}{x^2}+\frac{pr}{x}+ps$$
$$=ps-\frac{qr}{x^2}$$

Question 4:

Find the derivative of the following function (it is to be understood that a, b, c, d are fixed non-zero constants): $(ax+b)(cx+d)^2$

Solution:

Let $f'(x)=(ax+b)(cx+d)^2$

By Leibnitz product rule,

$$f'(x)=(ax+b)\frac{d}{dx}(cx+d)^2+(cx+d)^2\frac{d}{dx}(ax+b)$$
$$=(ax+b)\frac{d}{dx}\left(c^2x^2+2cdx^2+d^2\right)+(cx+d)^2\frac{d}{dx}(ax+b)$$
$$=(ax+b)\left[\frac{d}{dx}\left(c^2x^2\right)+\frac{d}{dx}(2cdx)+\frac{d}{dx}\left(d^2\right)\right]+(cx+d)^2\left[\frac{d}{dx}(ax)+\frac{d}{dx}(b)\right]$$
$$=(ax+b)\left(2c^2x+2cd\right)+(cx+d)^2a$$
$$=2c(ax+b)(cx+d)+a(cx+d)^2$$

Question 5:

Find the derivative of the following functions (it is to be understood that a,b,c,d are fixed non-zero constants): $\frac{ax+b}{cx+d}$

Solution:

Let $f(x)=\frac{ax+b}{cx+d}$

By quotient rule,

$$f'(x)=\frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$
$$=\frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}$$
$$=\frac{acx+ad-acx-bc}{(cx+d)^2}$$
$$=\frac{ad-bc}{(cx+d)^2}$$

Question 6:

Find the derivative of the following functions: $\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$

Solution:

Let $f(x)=\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{\frac{x+1}{x}}{\frac{x-1}{x}}=\frac{x+1}{x-1}$, where $x\neq 0$

By quotient rule

$$f'(x)=\frac{(x-1)\frac{d}{dx}(x+1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2}, x\neq 0,1$$

$$=\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}, x\neq 0,1$$

$$=\frac{x-1-x-1}{(x-1)^2}, x\neq 0,1$$

$$=\frac{-2}{(x-1)^2}, x\neq 0,1$$

Question 7:

Find the derivative of the following function (it is to be understood that a, b, c are fixed non-zero constants): $\frac{1}{ax^2+bx+c}$

Solution:

Let $f(x)=\frac{1}{ax^2+bx+c}$

By quotient rule,

$$f'(x)=\frac{(ax^2+bx+c)\frac{d}{dx}(1)-\frac{d}{dx}(ax^2+bx+c)}{(ax^2+bx+c)^2}$$

$$=\frac{(ax^2+bx+c)(0)-(2ax+b)}{(ax^2+bx+c)^2}$$

$$=\frac{-(2ax+b)}{(ax^2+bx+c)^2}$$

Question 8:

Find the derivative of the following function (it is to be understood that a, b, p, q, r are fixed non-zero constants): $\frac{ax+b}{px^2+qx+r}$

Solution:

Let $f(x)=\frac{ax+b}{px^2+qx+r}$

By quotient rule,

$$f'(x)=\frac{\left(px^2+qx+r\right)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}\left(px^2+qx+r\right)}{\left(px^2+qx+r\right)^2}$$

$$=\frac{\left(px^2+qx+r\right)(a)-(ax+b)(2px+q)}{\left(px^2+qx+r\right)^2}$$

$$=\frac{apx^2+aqx+ar-aqx-2bpx-bq-2apx^2}{\left(px^2+qx+r\right)^2}$$

$$=\frac{-apx^2-2bpx+ar-bq}{\left(px^2+qx+r\right)^2}$$

Question 9:

Find the derivative of the following function (it is to be understood that a, b, p, q, r are fixed non-zero constants): $\frac{px^2+qx+r}{ax+b}$

Solution:

Let $f(x)=\frac{px^2+qx+r}{ax+b}$

By quotient rule,

$$f'(x)=\frac{(ax+b)\frac{d}{dx}\left(px^2+qx+r\right)-\left(px^2+qx+r\right)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

$$=\frac{(ax+b)(2px+q)-\left(px^2+qx+r\right)(a)}{(ax+b)^2}$$

$$=\frac{2apx^2+aqx+2bpx+bq-apx^2-aqx-ar}{(ax+b)^2}$$

$$=\frac{apx^2+2bpx+bq-ar}{(ax+b)^2}$$

Question 10:

Find the derivative of the following function (it is to be understood that a, b are fixed non-zero constants): $\frac{a}{x^4}-\frac{b}{x^2}+\cos x$

Solution:

Let $f(x)=\frac{a}{x^4}-\frac{b}{x^2}+\cos x$

$$f'(x)=\frac{d}{dx}\left(\frac{a}{x^4}\right)-\frac{d}{dx}\left(\frac{b}{x^2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\frac{d}{dx}\left(x^{-4}\right)-b\frac{d}{dx}\left(x^{-2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\left(-4x^{-5}\right)-b\left(-2x^{-3}\right)+(-\sin x) \qquad \left[\because \frac{d}{dx}\left(x^n\right)=nx^{n-1} \text{ and } \frac{d}{dx}(\cos x)=-\sin x\right]$$

$$=\frac{-4a}{x^5}+\frac{2b}{x^3}-\sin x$$

Question 11:

Find the derivative of the following function: $4\sqrt{x}-2$

Solution:

Let $f(x)=4\sqrt{x}-2$

$$f'(x)=\frac{d}{dx}\left(4\sqrt{x}-2\right)$$

$$=\frac{d}{dx}\left(4\sqrt{x}\right)-\frac{d}{dx}(2)=4\frac{d}{dx}\left(x^{\frac{1}{2}}\right)-0$$

$$=4\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)=\left(2x^{\frac{-1}{2}}\right)$$

$$=\frac{2}{\sqrt{x}}$$

Question 12:

Find the derivative of the following functions (it is to be understood that a, b are fixed non-zero constants and n is integer): $(ax+b)^n$

Solution:

Let $f(x)=(ax+b)^n$

Accordingly, $f(x+h)=\{a(x+h)+b\}^n=(ax+ah+b)^n$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{(ax+ah+b)^n-(ax+b)^n}{h}$$
$$=\lim_{h\to 0}\frac{(ax+b)^n\left(1+\frac{ah}{ax+b}\right)^n-(ax+b)^n}{h}$$
$$=(ax+b)^n\lim_{h\to 0}\frac{1}{h}\left[\left\{1+n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)}{2}\left(\frac{ah}{ax+b}\right)^2+\ldots\right\}-1\right] \qquad (\text{Using binomial theorem})$$
$$=(ax+b)^n\lim_{h\to 0}\frac{1}{h}\left[n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)a^2h^2}{2(ax+b)^2}+\ldots\ldots(\text{Terms containing higher degree of } h)\right]$$
$$=(ax+b)^n\lim_{h\to 0}\left[\frac{na}{(ax+b)}+\frac{n(n-1)a^2h}{2(ax+b)^2}+\ldots..\right]$$
$$=(ax+b)^n\left[\frac{na}{(ax+b)}+0\right]$$
$$=na\frac{(ax+b)^n}{ax+b}$$
$$=na(ax+b)^{n-1}$$

Question 13:

Find the derivative of the following functions (it is to be understood that a,b,c,d are fixed non-zero constants and m and n are integers): $(ax+b)^n(cx+d)^m$

Solution:

Let $f(x)=(ax+b)^n(cx+d)^m$

By Leibnitz product rule,

$$f'(x)=(ax+b)^n\frac{d}{dx}(cx+d)^m+(cx+d)^m\frac{d}{dx}(ax+b)^n \qquad \ldots(1)$$

Now,

Let $f_1(x)=(cx+d)^m$

Accordingly, $f_1(x+h)=(cx+ch+d)^m$

$$f_1'(x)=\lim_{h\to 0}\frac{f_1(x+h)-f_1(x)}{h}$$

$$=\lim_{h\to 0}\frac{(cx+ch+d)^m-(cx+d)^m}{h}$$

$$=(cx+d)^m\lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{ch}{cx+d}\right)^m-1\right]$$

$$=(cx+d)^m\lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{mch}{(cx+d)}+\frac{m(m-1)}{2}\frac{c^2h^2}{(cx+d)^2}+\ldots\right)^m-1\right]$$

$$=(cx+d)^m\lim_{h\to 0}\frac{1}{h}\left[\frac{mch}{(cx+d)}+\frac{m(m-1)c^2h^2}{2(cx+d)^2}+\ldots(\text{Terms containing higher degree of } h)\right]$$

$$=(cx+d)^m\lim_{h\to 0}\left[\frac{mc}{(cx+d)}+\frac{m(m-1)c^2h}{2(cx+d)^2}+\ldots\right]$$

$$=(cx+d)^m\left[\frac{mc}{(cx+d)}+0\right]$$

$$=\frac{mc(cx+d)^m}{(cx+d)}$$

$$=mc(cx+d)^{m-1}$$

Hence, $\frac{d}{dx}(cx+d)^m=mc(cx+d)^{m-1}$ ...(2)

Similarly, $\frac{d}{dx}(ax+b)^n=na(ax+b)^{n-1}$ ...(3)

Therefore, from (1), (2) and (3), we obtain

$$f'(x)=(ax+b)^n\left\{mc(cx+d)^{m-1}\right\}+(cx+d)^m\left\{na(ax+b)^{n-1}\right\}$$

$$=(ax+b)^{n-1}(cx+d)^{m-1}\left[mc(ax+b)+na(cx+d)\right]$$

Question 14:

Find the derivative of the following function (it is to be understood that a is fixed non-zero constant): $\sin(x+a)$

Solution:

Let $f(x)=\sin(x+a)$

Accordingly, $f(x+h)=\sin(x+h+a)$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\sin(x+h+a)-\sin(x+a)}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left[\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]\right]$$
$$=\lim_{h\to 0}\cos\left(\frac{2x+2a+h}{2}\right)\lim_{\frac{h}{2}\to 0}\left[\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right] \qquad \left[\text{As } h\to 0\Rightarrow\frac{h}{2}\to 0\right]$$
$$=\cos\left(\frac{2x+2a}{2}\right)\times 1 \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$
$$=\cos(x+a)$$

Question 15:

Find the derivative of the following function: $\operatorname{cosec} x\cot x$

Solution:

Let $f(x)=\operatorname{cosec} x\cot x$

By Leibnitz product rule,

$$f'(x)=\operatorname{cosec} x(\cot x)'+\cot x(\operatorname{cosec} x)' \qquad \ldots(1)$$

Now,

Let $f_1(x)=\cot x$

Accordingly, $f_1(x+h)=\cot(x+h)$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\cot(x+h)-\cot(x)}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{\sin x\cos(x+h)-\cos x\sin(x+h)}{\sin x\sin(x+h)}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right)\\
&= \frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)}\right]\\
&= \frac{-1}{\sin x}\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\sin(x+h)}\right)\\
&= \frac{-1}{\sin x}.1.\left(\lim_{h\to 0}\frac{1}{\sin(x+0)}\right)\\
&= \frac{-1}{\sin^2 x}\\
&= -\operatorname{cosec}^2 x
\end{aligned}
$$

Therefore, $(\cot x)' = -\operatorname{cosec}^2 x \qquad \ldots(2)$

Now,

Let $f_2(x) = \operatorname{cosec} x$

Accordingly, $f_2(x+h) = \operatorname{cosec}(x+h)$

By first principle,

$$f_2'(x)=\lim_{h\to 0}\frac{f_2(x+h)-f_2(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec}(x)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\sin x-\sin(x+h)}{\sin x\sin(x+h)}\right)$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

$$f_2'(x)=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{-\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{-1}{\sin x}\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\lim_{h\to 0}\frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$=\frac{-1}{\sin x}.1.\frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$$

$$=\frac{-1}{\sin x}.\frac{\cos x}{\sin x}$$

$$=-\operatorname{cosec} x.\cot x$$

Therefore, $(\operatorname{cosec} x)'=-\operatorname{cosec} x.\cot x \qquad \ldots(3)$

From (1), (2) and (3)

$$f'(x)=\operatorname{cosec} x\left(-\operatorname{cosec}^2 x\right)+\cot x\left(-\operatorname{cosec} x\cot x\right)$$

$$=-\operatorname{cosec}^3 x-\cot^2 x\operatorname{cosec} x$$

Question 16:

Find the derivative of the following function: $\dfrac{\cos x}{1+\sin x}$

Solution:

Let $f(x)=\dfrac{\cos x}{1+\sin x}$

By quotient rule,

$$f'(x)=\frac{(1+\sin x)\frac{d}{dx}(\cos x)-(\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$=\frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$

$$=\frac{-\sin x-(\sin^2 x+\cos^2 x)}{(1+\sin x)^2}$$

$$=\frac{-\sin x-1}{(1+\sin x)^2}$$

$$=\frac{-(1-\sin x)}{(1+\sin x)^2}$$

$$=\frac{-1}{(1+\sin x)^2}$$

Question 17:

Find the derivative of the following function: $\dfrac{\sin x+\cos x}{\sin x-\cos x}$

Solution:

Let $f(x)=\dfrac{\sin x+\cos x}{\sin x-\cos x}$

By quotient rule,

$$f'(x)=\frac{(\sin x-\cos x)\frac{d}{dx}(\sin x+\cos x)-(\sin x+\cos x)\frac{d}{dx}(\sin x-\cos x)}{(\sin x-\cos x)^2}$$

$$=\frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$=\frac{-(\sin x-\cos x)^2-(\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

$$=\frac{-\left[\sin^2 x+\cos^2 x-2\sin x\cos x+\sin^2 x+\cos^2 x+2\sin x\cos x\right]}{(\sin x-\cos x)^2}$$

$$=\frac{-[1+1]}{(\sin x-\cos x)^2}$$

$$=\frac{-2}{(\sin x-\cos x)^2}$$

Question 18:

Find the derivative of the following function: $\frac{\sec x-1}{\sec x+1}$

Solution:

Let $f(x)=\frac{\sec x-1}{\sec x+1}=\frac{\frac{1}{\cos x}-1}{\frac{1}{\cos x}+1}=\frac{1-\cos x}{1+\cos x}$

By quotient rule,

$$f'(x)=\frac{(1+\cos x)\frac{d}{dx}(1-\cos x)-(1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)(\sin x)-(1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

$$=\frac{\sin x+\cos x\sin x+\sin x-\sin x\cos x}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{\left(1+\frac{1}{\sec x}\right)^2}$$

$$=\frac{2\sin x}{\frac{(\sec x+1)^2}{\sec^2 x}}$$

$$=\frac{2\sin x\sec^2 x}{(\sec x+1)^2}$$

$$=\frac{2\frac{\sin x}{\cos x}\sec x}{(\sec x+1)^2}$$

$$=\frac{2\sec x\tan x}{(\sec x+1)^2}$$

Question 19:

Find the derivative of the following function (it is to be understood that n is integer): $\sin^n x$

Solution:

Let $y=\sin^n x$

Accordingly, for $n=1,\ y=\sin x$

Therefore, $\frac{dy}{dx}=\cos x,\ i.e.,\ \frac{d}{dx}\sin x=\cos x$

For $n=2,\ y=\sin^2 x$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x \sin x)$$
$$= (\sin x)'(\sin x) + \sin x(\sin x)' \qquad \text{[By Leibnitz product rule]}$$
$$= \cos x \sin x + \sin x \cos x$$
$$= 2\sin x \cos x \qquad \ldots(1)$$

For $n = 3, y = \sin^3 x$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x \sin^2 x)$$
$$= (\sin x)' \sin^2 x + \sin x(\sin^2 x)' \qquad \text{[By Leibnitz product rule]}$$
$$= \cos x \sin^2 x + \sin x(2\sin x \cos x) \qquad \text{[using (1)]}$$
$$= \cos x \sin^2 x + \sin^2 x \cos x$$
$$= 3\sin^2 x \cos x$$

We assert that $\frac{d}{dx}(\sin^n x) = n\sin^{(n-1)} x \cos x$

Let our assertion be true for $n = k$

i.e., $\frac{d}{dx}(\sin^k x) = k\sin^{(k-1)} x \cos x \qquad \ldots(2)$

Consider,

$$\frac{d}{dx}(\sin^{k+1} x) = \frac{d}{dx}(\sin x \sin^{(k)} x)$$
$$= (\sin x)' \sin^k x + \sin x(\sin^k x)' \qquad \text{[By Leibnitz product rule]}$$
$$= \cos x \sin^k x + \sin x(k\sin^{k-1} x \cos x) \qquad \text{[using (2)]}$$
$$= \cos x \sin^k x + k\sin^k x \cos x$$
$$= (k+1)\sin^k x \cos x$$

Thus, our assertion is true for $n = k + 1$

Hence, by mathematical induction, $\frac{d}{dx}(\sin^n x) = n\sin^{(n-1)} x \cos x$

Question 20:

Find the derivative of the following functions (it is to be understood that a, b, c, d are fixed non-zero constants): $\frac{a+b\sin x}{c+d\cos x}$

Solution:

Let $f(x)=\frac{a+b\sin x}{c+d\cos x}$

By quotient rule,

$$f'(x)=\frac{(c+d\cos x)\frac{d}{dx}(a+b\sin x)-(a+b\sin x)\frac{d}{dx}(c+d\cos x)}{(c+d\cos x)^2}$$

$$=\frac{(c+d\cos x)(b\cos x)-(a+b\sin x)(-d\sin x)}{(c+d\cos x)^2}$$

$$=\frac{cb\cos x+bd\cos^2 x+ad\sin x+bd\sin^2 x}{(c+d\cos x)^2}$$

$$=\frac{bc\cos x+ad\sin x+bd\left(\cos^2 x+\sin^2 x\right)}{(c+d\cos x)^2}$$

$$=\frac{bc\cos x+ad\sin x+bd}{(c+d\cos x)^2}$$

Question 21:

Find the derivative of the following functions (it is to be understood that a is fixed non-zero constant): $\frac{\sin(x+a)}{\cos x}$

Solution:

Let $f(x)=\frac{\sin(x+a)}{\cos x}$

By quotient rule,

$$f'(x)=\frac{\cos x\frac{d}{dx}\left[\sin(x+a)\right]-\sin(x+a)\frac{d}{dx}\cos x}{\cos^2 x}$$

$$=\frac{\cos x\frac{d}{dx}\left[\sin(x+a)\right]-\sin(x+a)\frac{d}{dx}(-\sin x)}{\cos^2 x} \qquad \ldots(1)$$

Let $g(x)=\sin(x+a)$

Accordingly, $g(x+h)=\sin(x+h+a)$

By first principle,

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h+a)-\sin(x+a)\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right).\sin\left(\frac{h}{2}\right)\right]$$
$$=\lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]$$
$$=\lim_{h\to 0}\cos\left(\frac{2x+2a+h}{h}\right)\lim_{h\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\quad\left[\text{As } h\to 0\Rightarrow\frac{h}{2}\to 0\right]$$
$$=\left(\cos\frac{2x+2a}{2}\right)\times 1\qquad\left[\lim_{h\to 0}\frac{\sin h}{h}=1\right]$$
$$=\cos(x+a)\qquad\ldots(2)$$

From (1) and (2), we obtain

$$f'(x)=\frac{\cos x\cos(x+a)+\sin x\sin(x+a)}{\cos^2 x}$$
$$=\frac{\cos(x+a-x)}{\cos^2 x}$$
$$=\frac{\cos a}{\cos^2 x}$$

Question 22:

Find the derivative of the following function: $x^4(5\sin x-3\cos x)$

Solution:

Let $f(x)=x^4(5\sin x-3\cos x)$

By product rule,

$$f'(x) = x^4 \frac{d}{dx}(5\sin x - 3\cos x) + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$$
$$= x^4\left[5\frac{d}{dx}(\sin x) - 3\frac{d}{dx}(\cos x)\right] + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$$
$$= x^4\left[5\cos x - 3(-\sin x)\right] + (5\sin x - 3\cos x)(4x^3)$$
$$= x^3\left[5x\cos x + 3x\sin x + 20\sin x - 12\cos x\right]$$

Question 23:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(x^2+1)\cos x$

Solution:

Let $f(x) = (x^2+1)\cos x$

By product rule,

$$f'(x) = (x^2+1)\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(x^2+1)$$
$$= (x^2+1)(-\sin x) + \cos x(2x)$$
$$= -x^2\sin x - \sin x + 2x\cos x$$

Question 24:

Find the derivative of the following function (it is to be understood that a, p, q are fixed non-zero constants): $(ax^2+\sin x)(p+q\cos x)$

Solution:

Let $f(x) = (ax^2+\sin x)(p+q\cos x)$

By product rule,

$$f'(x) = (ax^2+\sin x)\frac{d}{dx}(p+q\cos x) + (p+q\cos x)\frac{d}{dx}(ax^2+\sin x)$$
$$= (ax^2+\sin x)(-q\sin x) + (p+q\cos x)(2ax+\cos x)$$
$$= -q\sin x(ax^2+\sin x) + (p+q\cos x)(2ax+\cos x)$$

Question 25:

Find the derivative of the following function: $(x+\cos x)(x-\tan x)$

Solution:

Let $f(x)=(x+\cos x)(x-\tan x)$

By product rule,

$$f'(x)=(x+\cos x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\cos x)$$

$$=(x+\cos x)\left[\frac{d}{dx}(x)-\frac{d}{dx}(\tan x)\right]+(x-\tan x)(1-\sin x)$$

$$=(x+\cos x)\left[1-\frac{d}{dx}(\tan x)\right]+(x-\tan x)(1-\sin x) \qquad \ldots(1)$$

Let $g(x)=\tan x$

Accordingly, $g(x+h)=\tan(x+h)$

By first principle,

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\cos(x+h)}\right)$$

$$=\frac{1}{\cos x}.1.\left(\frac{1}{\cos(x+0)}\right)$$

$$=\frac{1}{\cos^2 x}$$

$$=\sec^2 x \qquad \ldots(2)$$

Therefore, from (1) and (2), we obtain

$$f'(x)=(x+\cos x)(1-\sec^2 x)+(x-\tan x)(1-\sin x)$$
$$=(x+\cos x)(-\tan^2 x)+(x-\tan x)(1-\sin x)$$
$$=-\tan^2 x(x+\cos x)+(x-\tan x)(1-\sin x)$$

Question 26:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and *m* and *n* are integers): $\frac{4x+5\sin x}{3x+7\cos x}$

Solution:

Let $f(x)=\frac{4x+5\sin x}{3x+7\cos x}$

By quotient rule,

$$f'(x)=\frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x)-(4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$
$$=\frac{(3x+7\cos x)\left[4\frac{d}{dx}(x)+5\frac{d}{dx}(\sin x)\right]-(4x+5\sin x)\left[3\frac{d}{dx}(x)+7\frac{d}{dx}(\cos x)\right]}{(3x+7\cos x)^2}$$
$$=\frac{(3x+7\cos x)[4+5\cos x]-(4x+5\sin x)[3-7\sin x]}{(3x+7\cos x)^2}$$
$$=\frac{12x+15x\cos x+28\cos x-12x+28x\sin x-15\sin x+35(\cos^2 x+\sin^2 x)}{(3x+7\cos x)^2}$$
$$=\frac{15x\cos x+28\cos x+28x\sin x-15\sin x+35(\cos^2 x+\sin^2 x)}{(3x+7\cos x)^2}$$
$$=\frac{35+15x\cos x+28\cos x+28x\sin x-15\sin x}{(3x+7\cos x)^2}$$

Question 27:

Find the derivative of the following function: $\frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$

Solution:

Let $f(x)=\dfrac{x^2\cos\left(\dfrac{\pi}{4}\right)}{\sin x}$

By quotient rule,

$$f'(x)=\cos\left(\frac{\pi}{4}\right)\left[\frac{\sin x\dfrac{d}{dx}(x^2)-x^2\dfrac{d}{dx}(\sin x)}{\sin^2 x}\right]$$

$$=\cos\left(\frac{\pi}{4}\right)\left[\frac{\sin x(2x)-x^2(\cos x)}{\sin^2 x}\right]$$

$$=\frac{x\cos\frac{\pi}{4}[2\sin x-x\cos x]}{\sin^2 x}$$

Question 28:

Find the derivative of the following function: $\dfrac{x}{1+\tan x}$

Solution:

Let $f(x)=\dfrac{x}{1+\tan x}$

$$f'(x)=\frac{(1+\tan x)\dfrac{d}{dx}(x)-(x)\dfrac{dy}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$=\frac{(1+\tan x)-x\dfrac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \qquad \ldots(1)$$

Let $g(x)=1+\tan x$

Accordingly, $g(x+h)=1+\tan(x+h)$

By first principle,

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0}\left[\frac{1+\tan(x+h)-1-\tan x}{h}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos x\cos(x+h)}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right] \\
&= \left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right) \\
&= 1\times\frac{1}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x \qquad \ldots(2)
\end{aligned}
$$

From (1) and (2), we obtain

$$f'(x)=\frac{1+\tan x-x\sec^2 x}{(1+\tan x)^2}$$

Question 29:

Find the derivative of the following function: $(x+\sec x)(x-\tan x)$

Solution:

Let $f(x)=(x+\sec x)(x-\tan x)$

By product rule,

$$
\begin{aligned}
f(x) &= (x+\sec x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\sec x) \\
&= (x+\sec x)\left[\frac{d}{dx}(x)-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[\frac{d}{dx}(x)+\frac{d}{dx}\sec x\right] \\
&= (x+\sec x)\left[1-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[1+\frac{d}{dx}\sec x\right]
\end{aligned}
$$

Let $f_1(x)=\tan x,\ f_2(x)=\sec x$

Accordingly, $f_1(x+h)=\tan(x+h),\ f_2(x+h)=\sec(x+h)$

$$f_1'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$=\lim_{h\to 0}\left[\frac{\tan(x+h)-\tan x}{h}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$

$$=\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$=1\times\frac{1}{\cos^2 x}$$

$$=\frac{1}{\cos^2 x}$$

$$=\sec^2 x \qquad \ldots(2)$$

$$f_2'(x)=\lim_{h\to 0}\left(\frac{f_2(x+h)-f_2(x)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\sec(x+h)-\sec x}{h}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$f_2'(x)=\frac{1}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right).\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right).\sin\left(\frac{-h}{2}\right)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}\lim_{h\to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}}{\cos(x+h)}\right]$$

$$=\sec x\frac{\left\{\lim_{h\to 0}\sin\left(\frac{2x+h}{2}\right)\right\}\left\{\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}}{\lim_{h\to 0}\cos(x+h)}$$

$$=\sec x.\frac{\sin x.1}{\cos x}$$

$$=\sec x\tan x$$

Therefore, $\frac{d}{dx}(\sec x)=\sec x\tan x$

From (1), (2) and (3), we obtain

$$f'(x)=(x+\sec x)(1-\sec^2 x)+(x-\tan x)(1+\sec x\tan x)$$

Question 30:

Find the derivative of the following functions (it is to be understood that n is integer): $\frac{x}{\sin^n x}$

Solution:

Let $f(x)=\frac{x}{\sin^n x}$

By quotient rule,

$$f'(x)=\frac{\sin^n x\frac{d}{dx}x-x\frac{d}{dx}\sin^n x}{\sin^{2n}x}$$

It can be easily shown that $\frac{d}{dx}\sin^n x=n\sin^{n-1}x\cos x$

Therefore,

$$\begin{aligned}f'(x)&=\frac{\sin^n x\frac{d}{dx}x-x\frac{d}{dx}\sin^n x}{\sin^{2n}x}\\&=\frac{\sin^n x.1-x\left(n\sin^{n-1}x\cos x\right)}{\sin^{2n}x}\\&=\frac{\sin^{n-1}x\left(\sin x-nx\cos x\right)}{\sin^{2n}x}\\&=\frac{\sin x-nx\cos x}{\sin^{n+1}x}\end{aligned}$$